INTERSECTION PROPERTIES OF BALLS IN FINITE DIMENSIONAL ℓ_1 SPACES

OTTE HUSTAD

INTRODUCTION. In the paper [1], it was shown that a finite family $\mathcal{F}$ of closed discs in $\mathbb{C}$, say $\mathcal{F} = \{\{z : |z-z_j| \leq r_j\}\}_{j=1}^n$, has a non empty intersection if and only if

$$(1) \quad |\sum_{j=1}^n z_j u_j| \leq \sum_{j=1}^n r_j |u_j| ; \; u_1,\ldots,u_n \in \mathbb{C} \quad \text{and} \quad \sum_{j=1}^n u_j = 0 .$$

If we introduce the space

$$H^n(\mathbb{C}) = \{(u_j) \in \mathbb{C}^n : \sum_{j=1}^n u_j = 0\}$$

and equip this space with the norm

$$\|(u_j)\|_r = \sum_{j=1}^n r_j |u_j| ,$$

then it follows from the Krein-Milman theorem that (1) is valid if and only if

$$(2) \quad |\sum_{j=1}^n z_j u_j| \leq 1 \; ; \quad (u_j) \in \operatorname{Ext}(H^n(\mathbb{C}), \| \; \|_r)_1 ,$$

where $(H^n(\mathbb{C}), \| \; \|_r)_1$ denotes the closed unit ball in the space $(H^n(\mathbb{C}), \| \; \|_r)$. The relation (2) explains why it is of interest to find the extreme points of the unit ball in $(H^n(\mathbb{C}), \| \; \|_r)$, and it was shown in [1] that an element $u \in H^n(\mathbb{C})$ with $\|u\|_r = 1$ is an extreme point of the unit ball in $(H^n(\mathbb{C}), \| \; \|_r)$ if and only if the set of column vectors

$$(3) \quad \left\{ \begin{pmatrix} u_j \\ r_j |u_j| \end{pmatrix} : 1 \leq j \leq n \text{ and } |u_j| \neq 0 \right\} \subset \mathbb{C} \times \mathbb{R}$$

is linearly independent over $\mathbb{R}$. It follows, in particular, that if u is an extreme point, then at most three of the coordinates of u are different from zero. The Helly theorem for discs in $\mathbb{C}$ is an immediate consequence of this result.

Recently, it has been shown by Å. Lima [2] that the criteria (1) and (2) are of a very general character. In fact, let A be a Banach space over $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$), let A^* be the dual of A and let

$$H^n(A^*) = \{(f_j) \in (A^*)^n : \sum_{j=1}^{n} f_j = 0\} .$$

Equip this space with the norm

$$\|(f_j)\|_r = \sum_{j=1}^{n} r_j \|f_j\| ,$$

where $r = (r_j)$ is a given multi-radius. Then Lima proved by a separation argument the following lemma (we denote with $B(a,R)$ the closed ball with center a and radius R).

LEMMA (Lima). Let $\{B(a_j,r_j)\}_{j=1}^{n}$ be a given family of closed balls in A . Then the following two conditions are equivalent:

(i) $\bigcap_{j=1}^{n} B(a_j,r_j+\epsilon) \neq \emptyset$; $\epsilon > 0$.

(ii) $\left| \sum_{j=1}^{n} f_j(a_j) \right| \leq 1$; $f = (f_j) \in \mathrm{Ext}(H^n(A^*),\| \|_r)_1$.

(Here $(H^n(A^*),\| \|_r)_1$ denotes the closed unit ball and Ext denotes the set of extreme points).

By another separation argument, Lima also proved the

COROLLARY. Let A be finite dimensional (or more generally, a dual space), let $n > k \geq 2$. Then A has the n,k intersection property (as defined in [3]) if and only if for any multi-radius $r = (r_j)$ the extreme points of the unit ball in $(H^n(A^*), \| \|_r)$ have at most k components different from zero.

These results exhibit the close connection that exists between intersection properties of balls in A and the structure of the set of extreme points of the unit ball in $(H^n(A^*), \| \|_r)$. Other examples of this connection can be found in [1] and in [2].

In the present paper our main aim is to characterize the extreme points of the unit ball in $(H^n(\ell^m_\infty(\mathbb{K})), \| \|_r)$, and to use this characterization to study intersection properties of balls in $\ell^m_1(\mathbb{K})$.

THE EXTREME POINTS OF $(H^n(\ell^m_\infty(\mathbb{K})))_1$. **An element u in** $H^n(\ell^m_\infty(\mathbb{K}))$ is of the form $u = (u_1, \ldots, u_n)$, where $u_1, \ldots, u_n \in \ell^m_\infty(\mathbb{K})$ and where $\Sigma u_j = 0$. We can and shall write each u_j as a column vector

$$u_j = \begin{pmatrix} u_{1,j} \\ \vdots \\ u_{m,j} \end{pmatrix}$$

and the norm of u_j is then given by

$$\|u_j\| = \max\{|u_{i,j}| : 1 \leq i \leq m\} .$$

Hence we can and shall represent u as an (m,n) matrix $(u_{i,j})$ such that the sum of the elements in each line is zero, that is

$$\sum_{j=1}^{n} u_{i,j} = 0 , \quad i = 1, \ldots, m .$$

At this point we note that if there exist an $i_o \in \{1, \ldots, m\}$ and two different indices $j_1, j_2 \in \{1, \ldots, n\}$ such that $|u_{i_o,j_1}| < \|u_{j_1}\|$ and $|u_{i_o,j_2}| < \|u_{j_2}\|$, then u can not be an extreme point. In fact, if we choose $\epsilon > 0$ small and define $p = (p_{i,j})$ by putting $p_{i_o,j_1} = u_{i_o,j_1} + \epsilon$; $p_{i_o,j_2} = u_{i_o,j_2} - \epsilon$, and $p_{i,j} = u_{i,j}$ elsewhere, and if $q = (q_{i,j})$ is defined similarly only interchanging the role of ϵ and $-\epsilon$; then $p,q \in H^n(\ell^m_\infty(\mathbb{K}))$ and $\|p\|_r = \|q\|_r = \|u\|_r$ and we have $2u = p+q$.

Hence the following condition (S) must be satisfied if u is an extreme point of the unit ball in $(H^n(\ell^m_\infty(\mathbb{K})), \|\ \|_r)$:

(S) There exists a function

$$\varkappa : \{1,\dots,m\} \to \{1,\dots,n\}$$

such that

$$|u_{i,j}| = \|u_j\| \ ; \ i \in \{1,\dots,m\} \ , \quad j \neq \varkappa(i) \ .$$

Looking at the one dimensional case (i.e. $m = 1$), a fair guess could be that if $u \in (H^n(\ell^m_\infty(\mathbb{K})), \|\ \|_r)$ has norm one and satisfies (S), and the set of column vectors

$$\left\{ \begin{pmatrix} u_j \\ r_j \|u_j\| \end{pmatrix} : 1 \leq j \leq n \quad \text{and} \quad \|u_j\| \neq 0 \right\} \tag{4}$$

is linearly independent over $\mathbb{R}$, then u is an extreme point. However, the following example shows that this need not be true. Let

$$u = \begin{pmatrix} \frac{1}{3} , & -\frac{1}{3} , & 0 \\ 0 , & \frac{1}{3} , & -\frac{1}{3} \end{pmatrix}$$

Then $\|u\|_1 = 1$ and the condition (S) is satisfied. Furthermore, the set

$$\left\{ \begin{pmatrix} u_j \\ \|u_j\| \end{pmatrix} \right\}_{j=1}^{3}$$

is linearly independent over $\mathbb{R}$. But if we put

$$a = \begin{pmatrix} -\frac{1}{6} , & 0 , & \frac{1}{6} \\ \frac{1}{6} , & 0 , & -\frac{1}{6} \end{pmatrix}$$

and let $p = u + a$, $q = u - a$, then $\|p\|_1 = \|q\|_1 = 1$, and $u = \frac{1}{2}(p+q)$.

We shall now assume that $u \in H^n(\ell^m_\infty(\mathbb{K}))$ satisfies the condition (S) , and we define

$$I = I(u) = \{i : |u_{i,\varkappa(i)}| < \|u_{\varkappa(i)}\|\} .$$

For each $j \in \{1,\ldots,n\}$ we let $\tilde{u}_j$ denote the column vector obtained from u_j by deleting the element $u_{i,j}$ if $i \in I(u)$. Hence $\tilde{u}_j$ has m-s coordinates, where s is the cardinal number of $I(u)$. (We do not exlude the possibility that $\tilde{u}_j$ is the empty column vector; this is the case in the example above). Finally, we put

$$J = J(u) = \{j : \|u_j\| \neq 0\} .$$

We then have the following

THEOREM 1. Let $u \in (H^n(\ell^m_\infty(\mathbb{K})), \| \ \|_r)$ and assume that $\|u\|_r = 1$. Then u is an extreme point of the unit ball in $(H^n(\ell^m_\infty(\mathbb{K})), \| \ \|_r)$ if and only if u satisfies the condition (S) and the set of column vectors

$$\left\{ \begin{pmatrix} \tilde{u}_j \\ r_j\|u_j\| \end{pmatrix} : j \in J(u) \right\} \tag{5}$$

is linearly independent over $\mathbb{R}$.

The proof of this theorem is modelled after the proof in [1] of the case $m = 1$. The details are, however, considerably more involved in the general case. We shall first prove two lemmata, the first of which corresponds to [1, Lemma 3.2].

TWO LEMMATA.

LEMMA 1. Let $u \in (H^n(\ell_\infty^m(\mathbb{K})), \| \ \|_r)$ and assume $\|u\|_r = 1$ and that u satisfies the condition (S). Let $p,q \in H^n(\ell_\infty^m(\mathbb{K}))$ with $\|p\|_r, \|q\|_r \leq 1$ and assume that $u = \frac{1}{2}(p+q)$. Put $a = p-u$. Then there exist n real numbers $t_1,\ldots,t_n \in [-1,1]$ such that

$$(6) \quad a_{i,k} = \begin{cases} t_k u_{i,k} \,; & k \neq \varkappa(i) \\ -\sum\limits_{\ell \neq \varkappa(i)} t_\ell \, u_{i,\ell} \,; & k = \varkappa(i) \end{cases}$$

Furthermore,

$$(7) \quad \left\{ \begin{array}{l} |a_{i,\varkappa(i)} - u_{i,\varkappa(i)}| \leq \|u_{\varkappa(i)}\|(1 - t_{\varkappa(i)}) \\ |a_{i,\varkappa(i)} + u_{i,\varkappa(i)}| \leq \|u_{\varkappa(i)}\|(1 + t_{\varkappa(i)}) \end{array} \right. ; \ i \in \{1,\ldots,m\}$$

and

$$(8) \quad \sum_{k=1}^{n} t_k r_k \|u_k\| = 0 \ .$$

Assume conversely that $t_1,\ldots,t_n \in [-1,1]$, and that a is given by (6) and that (7) and (8) are valid. Let $p = u+a$, $q = u-a$. Then $p,q \in H^n(\ell_\infty^m(\mathbb{K}))$, $\|p\|_r, \|q\|_r \leq 1$ and $u = \frac{1}{2}(p+q)$.

PROOF. We have

$$1 = \|u\|_r = \tfrac{1}{2}\|p+q\|_r \leq \tfrac{1}{2}(\|p\|_r + \|q\|_r) \leq 1 \ ,$$

and so $\|p\|_r = \|q\|_r = 1$. Hence

$$2 = \|p\|_r + \|q\|_r = \sum_{k=1}^{n} r_k(\|u_k + a_k\| + \|u_k - a_k\|) =$$

$$= \sum_{k=1}^{n} 2 r_k \|u_k\| \ .$$

Since we always have the inequalities

$$(9)\qquad 2\|u_k\| \leq \|u_k + a_k\| + \|u_k - a_k\| , \quad k = 1,\dots,n ,$$

we get from the equation above

$$(10)\qquad \|u_k + a_k\| + \|u_k - a_k\| \leq 2\|u_k\| ; \quad k = 1,\dots,n .$$

By definition, this means that

$$(11)\qquad |u_{i,k} + a_{i,k}| + |u_{j,k} - a_{j,k}| \leq 2\|u_k\| ; \quad \begin{matrix} i,j \in \{1,\dots,m\} \\ k \in \{1,\dots,n\} \end{matrix} .$$

Let now $k \in \{1,\dots,n\}$ be given. Choose $i \in \{1,\dots,m\}$ such that

$$(12)\qquad |u_{i,k}| = \max_{1 \leq j \leq n} \{|u_{j,k}|\} = \|u_k\| .$$

If we choose $j = i$ in (11), we get

$$|u_{i,k} + a_{i,k}| + |u_{i,k} - a_{i,k}| \leq 2\|u_k\| = 2|u_{i,k}| \leq$$
$$\leq |u_{i,k} + a_{i,k}| + |u_{i,k} - a_{i,k}| .$$

It follows that $a_{i,k}$ is located on the degenerated ellipse with foci in $u_{i,k}$ and $-u_{i,k}$. Hence there exists a real number $t_k \in [-1,1]$ such that

$$(13)\qquad a_{i,k} = t_k u_{i,k} .$$

By (12) and (13) we get

$$(14)\qquad \begin{cases} |u_{i,k} + a_{i,k}| = (1+t_k)|u_{i,k}| = (1+t_k)\|u_k\| \\ |u_{i,k} - a_{i,k}| = (1-t_k)|u_{i,k}| = (1-t_k)\|u_k\| . \end{cases}$$

Let now $j \in \{1,\dots,m\}$ be given. If we apply (11) and the first equation in (14), we get

$$(15)(i) \qquad |u_{j,k} - a_{j,k}| \leq (1-t_k)\|u_k\|; \; j \in \{1,\dots,m\}, \; k \in \{1,\dots,n\},$$

and if we interchange the role of i and j in (11), we get from the second equation in (14)

$$(15)(ii) \qquad |u_{j,k} + a_{j,k}| \leq (1+t_k)\|u_k\|; \; j \in \{1,\dots,m\}, \; k \in \{1,\dots,n\}.$$

Now we have, according to the condition (S), that if $k \neq \varkappa(j)$, then $|u_{j,k}| = \|u_k\|$. But then it follows from (15)(i) and (15)(ii) that $a_{j,k}$ belongs to the intersection of the two balls with centra in $u_{j,k}$ and $-u_{j,k}$, and with radii $(1-t_k)|u_{j,k}|$ and $(1+t_k)|u_{j,k}|$. Hence it follows that

$$a_{j,k} = t_k u_{j,k}; \qquad k \neq \varkappa(j).$$

On the other hand, since $a = p - u \in H^n(\ell^m_\infty(\mathbb{K}))$, we have

$$a_{j,\varkappa(j)} = -\sum_{k \neq \varkappa(j)} a_{j,k} = -\sum_{k \neq \varkappa(j)} t_k u_{j,k}.$$

Thus we have proved (6). Furthermore, (7) follows at once from (15)(i) and (15)(ii). As for (8), we get from (15)(ii)

$$1 = \|p\|_r = \sum_{k=1}^{n} r_k \max_j\{|u_{j,k} + a_{j,k}|\} \leq \sum_{k=1}^{n} r_k (1+t_k)\|u_k\|$$

$$= \sum_{k=1}^{n} r_k\|u_k\| + \sum_{k=1}^{n} r_k t_k\|u_k\| = 1 + \sum_{k=1}^{n} r_k t_k \|u_k\|.$$

In a similar way, we get from (15)(i)

$$1 = \|q\|_r \leq 1 - \sum_{k=1}^{n} r_k t_k \|u_k\|.$$

This proves (8).

Let us conversely assume that $t_1,\dots,t_n \in [-1,1]$ are given and that a is defined by (6), and that (7) and (8) are valid.

Let $p = u + a$ and $q = u - a$. Since it follows from the definition of a that a is in $H^n(\ell_\infty^m(\mathbb{K}))$, we at once get that p and q are also in this space. It is obvious that $u = \frac{1}{2}(p + q)$. Therefore, we have only to prove that $\|p\|_r, \|q\|_r \leq 1$. By definition

$$p_{i,k} = \begin{cases} (1 + t_k)u_{i,k}, & k \neq \varkappa(i) \\ u_{i,k} + a_{i,k}, & k = \varkappa(i). \end{cases}$$

Hence it follows from (7) that

$$\|p_k\| = \max_i\{|p_{i,k}|\} \leq (1 + t_k)\|u_k\| .$$

In a similar way we get

$$\|q_k\| = \max_i\{|q_{i,k}|\} \leq (1 - t_k)\|u_k\| .$$

By (8) we therefore obtain

$$\|p\|_r = \Sigma r_k\|p_k\| \leq \Sigma r_k(1 + t_k)\|u_k\| = \Sigma r_k\|u_k\| = \|u\|_r = 1$$

and

$$\|q\|_r = \Sigma r_k\|q_k\| \leq \Sigma r_k(1 - t_k)\|u_k\| = \Sigma r_k\|u_k\| = 1 .$$

This finishes the proof of Lemma 1.

LEMMA 2. Let $\mathbf{u} \in (H^n(\ell_\infty^m(\mathbb{K})), \| \ \|_r)$ and let $\|u\|_r = 1$. If u satisfies the condition (S), and if the set

$$\left\{\begin{pmatrix} \tilde{u}_j \\ r_j\|u_j\| \end{pmatrix} : j \in J(u)\right\}$$

is linearly dependent over $\mathbb{R}$, then u is not an extreme point of the unit ball in $(H^n(\ell_\infty^m(\mathbb{K})), \| \ \|_r)$

PROOF. It follows from the definition of $I(u)$ that there exists a positive $\delta \leq 1$ with the property that if $t_j \in [-\delta,\delta]$, $j \in J = J(u)$, then

$$(16)\qquad \begin{cases} \left|\sum_{j\in J\setminus\varkappa(i)} t_j u_{i,j} + u_{i,\varkappa(i)}\right| \leq \|u_{\varkappa(i)}\|(1-t_{\varkappa(i)}) \ ; & i \in I \\ \left|\sum_{j\in J\setminus\varkappa(i)} t_j u_{i,j} - u_{i,\varkappa(i)}\right| \leq \|u_{\varkappa(i)}\|(1+t_{\varkappa(i)}) \ ; & i \in I. \end{cases}$$

Now, by assumption, there exists a set $\{t_j \colon j \in J(u)\} \subset \mathbb{R}$ such that

$$(17)\qquad \sum_{j\in J} t_j u_{i,j} = 0 \ ; \qquad i \in \{1,\dots,m\}\setminus I$$

and

$$(18)\qquad \sum_{j\in J} t_j r_j \|u_j\| = 0 \ ,$$

and such that at least one $t_j \neq 0$. By dividing (17) and (18) with $\delta^{-1}\max\{|t_j|\}$, we can and shall assume that every $t_j \in [-\delta,\delta]$. Hence (16) is also valid. Put $t_j = 0$ if $j \in \{1,\dots,n\}\setminus J$, and define $a = (a_{i,j})$ by the equations (6) in Lemma 1. We then note that it follows from (17) that if $i \notin I$, then

$$-a_{i,\varkappa(i)} = \sum_{j\neq\varkappa(i)} t_j u_{i,j} = -t_{\varkappa(i)} u_{i,\varkappa(i)} \ .$$

This means that

$$(19)\qquad a_{i,\varkappa(i)} = t_{\varkappa(i)} u_{i,\varkappa(i)} \ , \qquad i \notin I.$$

It follows that

$$(20)\qquad \begin{cases} |a_{i,\varkappa(i)} - u_{i,\varkappa(i)}| \leq (1-t_{\varkappa(i)})\|u_{\varkappa(i)}\| \\ |a_{i,\varkappa(i)} + u_{i,\varkappa(i)}| \leq (1+t_{\varkappa(i)})\|u_{\varkappa(i)}\| \end{cases} ; \qquad i \notin I$$

We now observe that by the definition of a, the inequalities (16) can be written

$$(21)\qquad \begin{cases} |a_{i,\varkappa(i)} - u_{i,\varkappa(i)}| \leq (1 - t_{\varkappa(i)})\|u_{\varkappa(i)}\| \\ \\ |a_{i,\varkappa(i)} + u_{i,\varkappa(i)}| \leq (1 + t_{\varkappa(i)})\|u_{\varkappa(i)}\| \end{cases} ; \quad i \in I$$

It follows from (20) and (21) that the condition (7) in Lemma 1 is satisfied. Furthermore, since (8) is a consequence of (18), we have, by Lemma 1, that $u = \frac{1}{2}(p+q)$, where $p,q \in (H^n(\ell_\infty^m(\mathbb{K}), \| \ \|_r)$ are such that $\|p\|_r, \|q\|_r \leq 1$ and $p = u + a$, $q = u - a$. Hence we have only to show that $a \neq 0$. To achieve this, we reason as follows: From (18) we get that there exist at least two different elements $j,k \in J(u)$ such that $t_j \neq 0$ and $t_k \neq 0$. Choose $i \in \{1,\ldots,m\}$ such that $|u_{i,j}| = \|u_j\| > 0$. If $j \neq \varkappa(i)$, then it follows from (6) that $a_{i,j} = t_j u_{i,j} \neq 0$. On the other hand, if $j = \varkappa(i)$, then we get by the definition of $\varkappa$ that

$$|u_{i,\ell}| = \|u_\ell\| , \quad \ell \in \{1,\ldots,m\} \setminus \{j\} .$$

Since $k \in J(u)$, we get in particular $|u_{i,k}| = \|u_k\| > 0$. And since $k \neq j = \varkappa(i)$, it follows that $a_{i,k} = t_k u_{i,k} \neq 0$.

PROOF OF THEOREM 1.

Let u be an extreme point of the unit ball in $(H^n(\ell_\infty^m(\mathbb{K})), \| \ \|_r)$. Then we have already seen that u must satisfy (S), and hence it follows from Lemma 2 that the set

$$(22)\qquad \left\{ \begin{pmatrix} \tilde{u}_j \\ r_j \|u_j\| \end{pmatrix} : j \in J(u) \right\}$$

is linearly independent over $\mathbb{R}$. Assume conversely that this condition is fulfilled and that u satisfies (S). If u is not an extreme point, then it follows from Lemma 1 that there exists an $a \in H^n(\ell_\infty^m(\mathbb{K}))$ given by (6) such that $a \neq 0$ and such that (7) and (8) are valid. Since $a \neq 0$, there exists $k \in J(u)$ such that $t_k \neq 0$. Now let $i \in \{1,\dots,m\} \setminus I$. Then, by the definition of I, $|u_{i,\varkappa(i)}| = \|u_{\varkappa(i)}\|$. It follows from (7) that

$$a_{i,\varkappa(i)} = t_{\varkappa(i)} u_{i,\varkappa(i)} .$$

From (6) we therefore get

$$0 = \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} t_j u_{i,j} = \sum_{j\in J} t_j u_{i,j} \; ; \quad i \notin I .$$

Since it follows from (8) that

$$\sum_{j\in J} t_j r_j \|u_j\| = 0 ,$$

and since we know that $t_k \neq 0$ where $k \in J$, we have got that the set (22) can not be linearly independent over $\mathbb{R}$. This contradiction shows that u must be an extreme point of the unit ball in $(H^n(\ell_\infty^m(\mathbb{K})), \| \ \|_r)$.

As an immediate consequence of Theorem 1 we get the following

COROLLARY. Let $u = (u_j)$ be an extreme point of the unit ball in $(H^n(\ell_\infty^m(\mathbb{K})), \| \ \|_r)$. If $\mathbb{K} = \mathbb{C}$, then at most $2m+1$ components of u are different from zero, and if $\mathbb{K} = \mathbb{R}$, then at most $m + 1$ components of u are different from zero.

INTERSECTION OF BALLS IN $\ell_1^m(\mathbb{K})$. If we combine the corollary of Theorem 1 with the corollary of Limas Lemma, then we get an immediate proof of the Helly theorem for balls in $\ell_1^m(\mathbb{K})$. We shall now show that the Helly theorem is the best possible for the balls in $\ell_1^m(\mathbb{K})$. More precisely, we have the following

THEOREM 2. If $3 \leq n \leq 2m+1$, then $\ell_1^m(\mathbb{C})$ has not the $n,n-1$ intersection property. And if $3 < n \leq m+1$, then $\ell_1^m(\mathbb{R})$ has not the $n,n-1$ intersection property. However, $\ell_1^m(\mathbb{R})$ has the 3,2 intersection property.

COMMENT. It is well known that the last statement in Theorem 2 is true, see [3, Theorem 4.6] .

PROOF. According to the corollary of Limas Lemma, we have only to prove that the unit ball in $(H^n(\ell_\infty^m(\mathbb{K})), \| \|_1)$ admits an extreme point with all the n components different from zero.

<u>We assume first that $\mathbb{K} = \mathbb{C}$</u> . Let $\varphi = 2\pi/n$ and $\omega = e^{i\varphi}$. Define $k = [\frac{n}{2}]$ to be the integer value of $\frac{n}{2}$, and let the (m,n) matrix u be defined by

$$u = \frac{1}{n}\left\{\begin{matrix} \omega^1 & , \omega^2 & ,\ldots, & \omega^n \\ \omega^{2\cdot 1} & , \omega^{2\cdot 2} & ,\ldots, & \omega^{2n} \\ \vdots & \vdots & & \vdots \\ \omega^k & , \omega^{k2} & ,\ldots, & \omega^{kn} \\ \omega^1 & , \omega^2 & ,\ldots, & \omega^n \\ \vdots & \vdots & & \vdots \\ \omega^1 & , \omega^2 & ,\ldots, & \omega^n \end{matrix}\right\}$$

Then $u \in H^n(\ell_\infty^m(\mathbb{C}))$ and $\|u\|_1 = 1$. We also note that all the elements of u have absolute value one. We claim that u is an extreme point of the unit ball in $(H^n(\ell_\infty^m(\mathbb{C})), \| \|_1)$. In fact, according to Theorem 1, we have only to prove that the matrix

$$M = \left\{\begin{matrix} \cos\varphi, & \cos 2\varphi, & \ldots, & \cos n\varphi \\ \sin\varphi, & \sin 2\varphi, & \ldots, & \sin n\varphi \\ \vdots & \vdots & & \vdots \\ \cos k\varphi, & \cos 2k\varphi, & \ldots, & \cos nk\varphi \\ \sin k\varphi, & \sin 2k\varphi, & \ldots, & \sin nk\varphi \\ 1, & 1, & \ldots, & 1 \end{matrix}\right\}$$

has rank n. But if n is an odd number, then it follows from a formula of R.F. Scott [4] that M has rank n. And if n is an even number, then we get the same conclusion from a formula of K. Weihrauch [5].

<u>The case $\mathbb{K} = \mathbb{R}$</u>. Let the (m,n) matrix u be given by

$$u = \frac{1}{2n-4}\left\{\begin{matrix} 1, & -1 & -1, & \ldots, & -1, & n-3 \\ -1, & 1, & -1, & \ldots, & -1, & n-3 \\ \vdots & & & \ddots & \vdots & \vdots \\ -1, & \ldots & & ; 1, & -1, & n-3 \\ -1, & \ldots & & ,-1, & 1, & n-3 \\ \vdots & & & \vdots & \vdots & \vdots \\ -1, & \ldots & & ,-1, & 1, & n-3 \end{matrix}\right\}$$

If we can show that the (n,n) determinant

$$D = \begin{vmatrix} 1, & -1, & \ldots, & -1 & -1, & n-3 \\ \vdots & & & \vdots & & \vdots \\ -1, & \ldots & & , -1, & 1, & n-3 \\ 1, & \ldots & & , 1, & 1, & n-3 \end{vmatrix}$$

is different from zero, then it would follow from Theorem 1 that u is an extreme point of the unit ball in $(H^n(\ell_\infty^m(\mathbb{R})), \| \ \|_1)$. But adding the last line of D to the other ones we get

$$D = 2^{n-1}(n-3)\begin{vmatrix} 1, & 0, & \dots & & , 0, & 1 \\ 0, & 1, & 0, \dots, & & 0, & 1 \\ \vdots & & & & \vdots & \vdots \\ 0, & \dots, & & 0, & 1, & 1 \\ 1, & 1, & \dots & 1, & 1, & 1 \end{vmatrix}$$

Expanding this determinant, we get

$$D = 2^{n-1}(n-3)(2-n) \neq 0 \ .$$

Finally, to prove that $\ell_1^m(\mathbb{R})$ has the 3,2 intersection property, it suffices in view of Theorem 1 to observe that if

$$u = \begin{Bmatrix} \epsilon_{1,1}t_1, & \epsilon_{1,2}t_2, & \epsilon_{1,3}t_3 \\ \epsilon_{2,1}t_1, & \epsilon_{2.2}t_2, & \epsilon_{2,3}t_3 \end{Bmatrix} \in H^3(\ell_\infty^2(\mathbb{R})) \ ,$$

where $t_1, t_2, t_3 > 0$ and $\epsilon_{i,j} = \pm 1$, then the two lines in u are linearly dependent.

REFERENCES.

[1] Hustad, O. Intersection properties of balls in complex Banach spaces whose duals are L_1 spaces, Acta Mathematica 132 (1974) 283-313.

[2] Lima, Å. Intersection properties of balls and subspaces in Banach spaces, Preprint Series, Inst. Math., University of Oslo No. 3 (1974).

[3] Lindenstrauss, J. Extension of compact operators, Mem. Amer. Math. Soc., 48 (1964).

[4] Scott, R.F. Note on a theorem of Professor Cayley's, Messenger of Math., 8 (1879) 155-157.

[5] Weihrauch, K. Über gewisse Determinanten, Zeitschrift f. Math. u. Phys., 33 (1887) 126-128.

UNIVERSITY OF OSLO, NORWAY